A Study of Compressible Potential and Asymptotic Viscous Flows for Corner Region

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A new and direct approach for calculating the first higher order potential flow along an axial corner is presented. For the incompressible potential flow, the present approach demonstrates that the displacement effects in the corner may be visualized as the superposition of the displacement effects for the two intersecting semi-infinite plates forming the corner. The compressible subsonic potential flow is then obtained by the Prandtl-Glauert rule. Linearized airfoil theory is used to determine the potential flow for the supersonic case. The asymptotic viscous flow, to lowest order, for the corner problem has been calculated for general compressible flow. The analysis presented here recovers all the previously obtained lowest order asymptotic solutions. Cross flow velocity profiles have been given for M_∞ between 0.001 to 4 for an adiabatic wall as well as for a prescribed temperature at the wall. The behavior of the cross flow skin-friction coefficient is shown to be quite different from that of the skin friction coefficient due to classical axial flow.

Introduction

THE boundary-region problem of flow along an axial corner has long been a challenging problem for the fluid dynamics analyst. Interest in this problem has recently grown considerably due to development of the space shuttle and other high speed flight vehicles. There have been several experimental studies of this problem. But a consistent analytical treatment seems to have emerged only after Rubin's[1] formulation of the incompressible corner flow problem for which numerical solutions were later calculated by Rubin and Grossman.[2]

Generalization of the incompressible corner flow analysis has been made by Weinberg and Rubin[3] who studied the corresponding compressible flow for a model fluid, i.e., $Pr = 1.0$ and $\mu \sim T$. The case of the general compressible corner flow has been recently analyzed by K. Ghia and Davis[4] who have also obtained solutions for the incompressible problem using an implicit numerical technique that seems to hold much promise for complicated problems of this type.

The specification of boundary conditions in all the above analyses for the corner flow requires the knowledge of, first of all, the zeroth-order potential flow along the corner and the classical boundary-layer solution over a 2-D flat plate. These solutions are well known even for general compressible fluid. Furthermore, the corner layer is primarily a situation of interacting boundary layers. It is, therefore, necessary to determine the first higher order potential flow in the corner as well as the resulting secondary cross flow in the boundary layer.

The first-order potential flow along the corner is governed by the three-dimensional Laplace equation. Rubin[1] determined this flow for the incompressible case by solving the governing differential equation with the help of suitable Green's functions. The solution for the case of a compressible model fluid is obtainable directly from the incompressible results and was employed by Weinberg and Rubin[3] in their solution of the compressible corner flow.

The present paper describes a different and considerably simpler approach for calculating the first-order potential flow for the corner problem for the case of a general compressible fluid. First, the incompressible potential flow solution is derived by considering it as a superposition of the potential solutions due to the displacement effects on the two flat plates comprising the corner. The compressible subsonic potential flow is then obtained by the Prandtl-Glauert similarity rule. Linearized airfoil theory is used to determine the potential flow for the supersonic case.

The first-order potential solution provides the necessary outer boundary condition for the secondary cross flow in the boundary layers on the two intersecting flat plates. This cross flow has been calculated by Libby,[5] Bloom,[6] and Weinberg and Rubin[3] for the model fluid. The results of the present paper make it now possible to determine this secondary flow for the general compressible fluid. In fact, calculations have been made for several freestream Mach number values and wall temperature conditions.

Potential Flow Solutions for Corner Problem

The corner flow geometry is shown in Fig. 1. The potential flow solution in region I consists of the uniform freestream and the flow due to displacement thickness on the intersecting flat plates. All flow variables have been nondimensionalized with respect to their corresponding freestream values, e.g., U, V, W by U_∞, ρ by ρ_∞, T by T_∞ and the x, y, z coordinates are nondimensionalized with respect to the characteristic length used in defining a Reynolds number for the problem.

Incompressible Potential Flow

For a general three-dimensional flow, the velocity vector can be represented in terms of two scalar stream functions (See Karamcheti,[7]) as

$$\rho \bar{V} = \operatorname{grad} \psi_1 \times \operatorname{grad} \psi_2 \tag{1}$$

where ρ is the density of the fluid, $\bar{V}$ is the velocity vector, and ψ_1 and ψ_2 are the two stream functions whose curves of intersection define the streamlines of the flow. All quantities in Eq. (1) are, in general, functions of the local position vector $\bar{r}$. However, for incompressible flow, $\rho(\bar{r}) \equiv 1$.

The outer inviscid expansion for the flow variables is of the following form, in terms of the small parameters ε_i

$$\psi_1 = \psi_{11} + \varepsilon_1 \psi_{12} + \varepsilon_2 \psi_{13} + \dots$$
$$\psi_2 = \psi_{21} + \varepsilon_1 \psi_{22} + \varepsilon_2 \psi_{23} + \dots$$
$$U = U_o + \varepsilon_1 U_1 + \varepsilon_2 U_2 + \dots \qquad V = V_o + \varepsilon_1 V_1 + \varepsilon_2 V_2 + \dots$$

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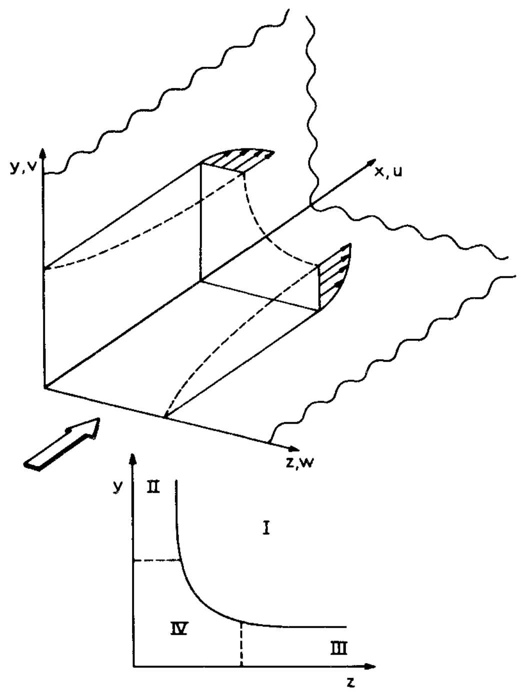

Fig. 1 Corner flow configuration.

and

$$W = W_o + \varepsilon_1 W_1 + \varepsilon_2 W_2 + \ldots \tag{2}$$

where

$$\varepsilon_{i+1} \ll \varepsilon_i \ll 1 \quad \text{and} \quad \varepsilon_1 = 1/(Re)^{1/2}$$

Since the axial corner is formed by the intersection of two semi-infinite flat plates, it is possible, according to linear theory, to consider one of the stream functions, say ψ_1, as being due to the flow over the plate $y = 0$, and the second stream function ψ_2 to describe the flow over the plate $z = 0$. Then, the first terms in the expansions (2) for the stream functions correspond to uniform flow over the plates. Further, the second terms ψ_{12} and ψ_{22} in the stream function expansions represent flow due to displacement thickness over the flat plates. Van Dyke[8] has given the stream function due to displacement effects for the semi-infinite flat plate. Then, since the displacement flow problem is linear, the stream functions ψ_1 and ψ_2 are obtained, up to second order, as

$$\psi_1(x,y) = y - 1/(Re)^{1/2}\beta[x+(x^2+y^2)^{1/2}]^{1/2} \tag{3}$$

$$\psi_2(x,z) = z - 1/(Re)^{1/2}\beta[x+(x^2+z^2)^{1/2}]^{1/2} \tag{4}$$

where β is a constant related to the displacement thickness δ_1 for the flat plate as

$$\beta = \delta_1/(2x)^{1/2} \tag{5}$$

For incompressible flow, it can be shown that $\delta_1 = 1.7208(x)^{1/2}$, so that $\beta = 1.21678$ for this case. Using Equations (3) and (4) in Eq. (1) yields the velocity vector. Thus, for incompressible flow, with $\rho(\bar{r}) \equiv 1$

$$\bar{V} = \bar{i}\left[\frac{\partial\psi_1}{\partial y}\frac{\partial\psi_2}{\partial z}\right] + \bar{j}\left[-\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial z}\right] + \bar{k}\left[-\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_1}{\partial y}\right] \tag{6}$$

Retaining terms only up to ε_1, i.e., $1/(Re)^{1/2}$, this leads to the solution for the zeroth-order potential flow for the incompressible corner problems as $U_o \equiv 1$, $V_o \equiv W_o \equiv 0$. Also, the velocity components for the first-order potential flow are obtained as

$$U_1 = -\frac{\beta}{2}\left[\frac{y(x^2+y^2)^{-1/2}}{[x+(x^2+y^2)^{1/2}]^{1/2}} + \frac{z(x^2+z^2)^{-1/2}}{[x+(x^2+z^2)^{1/2}]^{1/2}}\right] \tag{7}$$

$$V_1 = \frac{\beta}{2}\left[\frac{[x+(x^2+y^2)^{1/2}]^{1/2}}{(x^2+y^2)^{1/2}}\right] \tag{8}$$

$$W_1 = \frac{\beta}{2}\left[\frac{[x+(x^2+z^2)^{1/2}]^{1/2}}{(x^2+z^2)^{1/2}}\right] \tag{9}$$

The results obtained above are the same as those determined by Rubin,[1] although the algebra involved is considerably simpler here. Moreover, the present approach also demonstrates that the displacement effects in the corner may be visualized as the superposition of the displacement effects for the two intersecting semi-infinite plates forming the corner. These ideas can be extended to more complicated corner flow problems. The calculation of the first-order potential flow velocities for compressible flow is discussed next.

Subsonic Flow

The Prandtl-Glauert similarity rule provides a relation between the compressible potential flow velocity field and the corresponding incompressible flow as follows. The linearized theory for compressible flow will lead to the relations

$$y = (1-M_\infty{}^2)^{1/2} y_c = m y_c, \qquad z = (1-M_\infty{}^2)^{1/2} z_c = m z_c \tag{10}$$

and

$$C_{p_c} = (1/m) C_p \tag{11}$$

where the subscript c denotes quantities for the compressible flow.

Then, the similarity relations for the velocities can be derived as

$$U_{1,c} = (1/m) U_1, \quad V_{1,c} = V_1 \quad \text{and} \quad W_{1,c} = W_1 \tag{12}$$

Substituting for the incompressible velocities U_1, V_1 and W_1 from Eqs. (7–9) and for y and z from Eq. (10) into the relations given by Eq. (12) yields the velocities for the first order subsonic potential flow over the axial corner.

$$U_{1,c} = -\frac{\beta}{2^m}\left[\frac{[(x^2+m^2y^2)^{1/2}-x]^{1/2}}{(x^2+m^2y^2)^{1/2}} + \frac{[(x^2+m^2z^2)^{1/2}-x]^{1/2}}{(x^2+m^2z^2)^{1/2}}\right] \tag{13}$$

$$V_{1,c} = \frac{\beta}{2}\left[\frac{[(x^2+m^2y^2)^{1/2}+x]^{1/2}}{(x^2+m^2y^2)^{1/2}}\right] \tag{14}$$

$$W_{1,c} = \frac{\beta}{2}\left[\frac{[(x^2+m^2z^2)^{1/2}+x]^{1/2}}{(x^2+m^2z^2)^{1/2}}\right] \tag{15}$$

The two-dimensional analog of the above velocity expressions compares with the results of Maslen[9] for the second approximation to the laminar compressible boundary-layer flow over a semi-infinite flat plate. It must be borne in mind that, for the two-dimensional case, the above results would lead to a homogeneous boundary condition for the second order viscous velocity u. This is not true for the three-dimensional corner layer flow. As Eqs. (13–15) show, the first-order potential flow velocities for $y \to 0$ depend on z, and vice versa, and are not all zero in general. The velocity u will be zero only for $y \to 0$ and $z \to 0$ simultaneously.

For the compressible flow velocities given by Eqs. (13–15), the plates since the second-order problem is linear. Accordingly, using linear airfoil theory leads to the following two stream functions ψ_1 and ψ_2, up to second order, for the corner flow.

Supersonic Flow

The inviscid corner flow problem is again visualized as the superposition of the potential flow past two semi-infinite flat plates since the second-order problem is linear. Accordingly, using linear airfoil theory leads to the following two stream functions ψ_1 and ψ_2, up to second order, for the corner flow.

$$\psi_1(x,y) = y - [1/(Re)^{1/2}]\tau(x-\lambda y) \tag{16}$$

$$\psi_2(x,z) = z - [1/(Re)^{1/2}]\tau(x-\lambda z) \tag{17}$$

where τ is a thickness function. For the present problem, τ will be the displacement thickness for boundary-layer flow over a semi-infinite flat plate defined as follows for compressible flow.

$$\tau(x-\lambda y) = \beta(2)^{1/2}(x-\lambda y)^{1/2} = \beta(2)^{1/2}[x-(M_\infty{}^2-1)^{1/2}y]^{1/2} \tag{18}$$

where

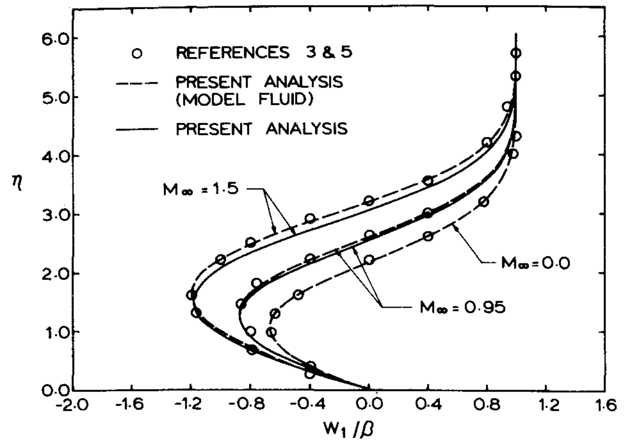

Fig. 2a Effect of "Model Fluid" assumption on cross flow velocity, adiabatic wall.

$$\beta = \int_o^\infty [1-(u_o/T_o)]\, d\eta \tag{19}$$

In Eq. (19), u_o and T_o are the classical nondimensional boundary-layer velocity and temperature, respectively, and η is the similarity coordinate. The derivation of Eq. (19) uses the definition of the displacement thickness δ_1 followed by a similarity transformation.

The velocity components can now be determined by substituting these stream functions [Eqs. (16) and (17)] into Eq. (1). Therefore, for compressible flow

$$(\rho_o+\varepsilon\rho_1+\cdots)(\bar{V}_o+\varepsilon\bar{V}_1+\cdots) = \bar{i}\left[\frac{\partial\psi_1}{\partial y}\frac{\partial\psi_2}{\partial z}\right]+ \bar{j}\left[-\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial z}\right]+\bar{k}\left[-\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_1}{\partial y}\right] \tag{20}$$

In nondimensional variables, $\rho_o = 1$, $\bar{V}_o = \bar{i}$. Then, if the first-order density ρ_1 is known, the first-order velocity $\bar{V}_1$ can be determined from Eq. (20). To calculate ρ_1, use is made of the kinematic form of the energy equation, the definition of the speed of sound, a, in terms of temperature for a perfect gas and the isentropic flow relation $\rho = T^{1/(\gamma-1)}$ to obtain

$$\rho_1 = -M_\infty{}^2 U_1 \tag{21}$$

Substituting Eq. (21) and Eqs. (16–18) into Eq. (20) finally yields the first-order velocities for supersonic potential flow along the corner.

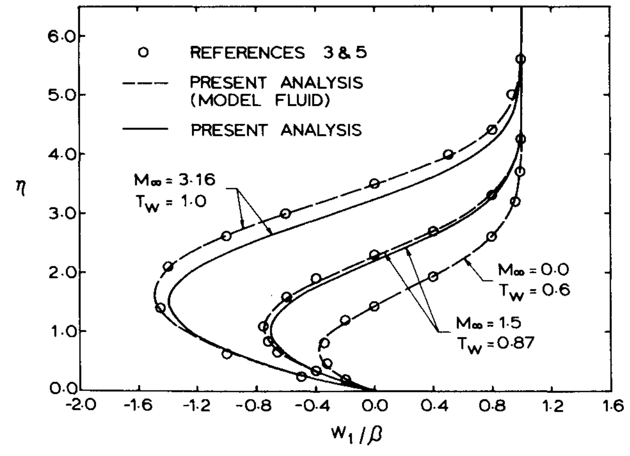

Fig. 2b Effect of "Model Fluid" assumption on cross flow velocity, cold wall.

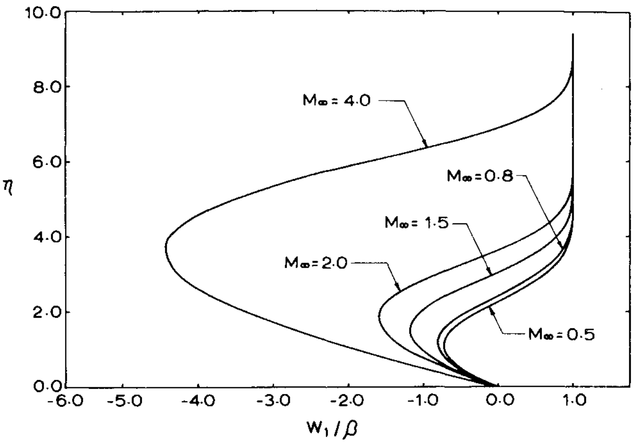

Fig. 3a Asymptotic cross flow velocity, adiabatic wall.

$$U_1 = -\frac{\beta}{(2)^{1/2}}\frac{1}{\lambda}\left[\frac{1}{(x-\lambda y)^{1/2}}+\frac{1}{(x-\lambda z)^{1/2}}\right] \tag{22}$$

$$V_1 = \frac{\beta}{(2)^{1/2}}\frac{1}{(x-\lambda y)^{1/2}} \tag{23}$$

$$W_1 = \frac{\beta}{(2)^{1/2}}\frac{1}{(x-\lambda z)^{1/2}} \tag{24}$$

Secondary Cross Flow in Boundary Layers

The first higher order potential flow determined in Sec. 2 provides the pressure gradient and the outer boundary condition for the secondary cross flow in the interacting boundary layers on the flat plates forming the corner. For the plate at $y = 0$, the cross flow w_1 is governed by the momentum equation for the z-direction, to lowest order, i.e.,

$$\rho_o\left[u_o\frac{\partial w_1}{\partial x}+v_1\frac{\partial w_1}{\partial Y}\right] = -\frac{\partial p_1}{\partial z}+\frac{\partial}{\partial Y}\left[\mu_o\frac{\partial w_1}{\partial Y}\right] \tag{25}$$

where u_o, v_1, μ_o and ρ_o comprise the classical boundary-layer solution for compressible flow; the independent coordinate y has been stretched such that $y = YRe^{-1/2}$ (Ref. 10). The pressure gradient for the viscous flow is determined from the matching condition at the outer edge of the viscous region and utilizing Eqs. (9), (15) and (24).

$$\left.\frac{\partial p_1}{\partial z}\right|_{\substack{Y\to\infty\\ z=0}} = -\left.\frac{\partial W_1}{\partial x}(x,z)\right|_{z=0} = +\frac{\beta}{2x(2x)^{1/2}} \tag{26}$$

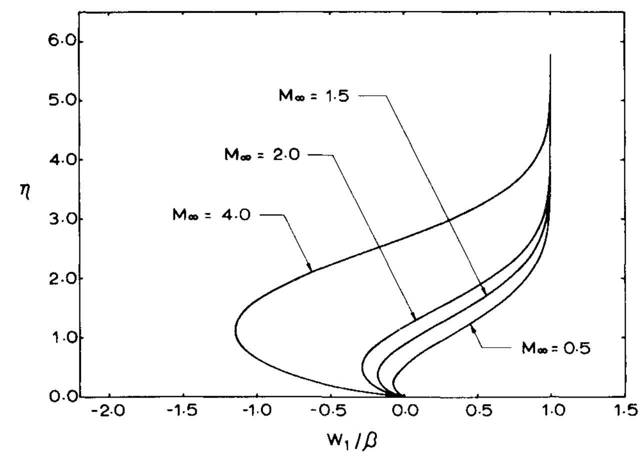

Fig. 3b Asymptotic cross flow velocity, $T_w = 0.25$.

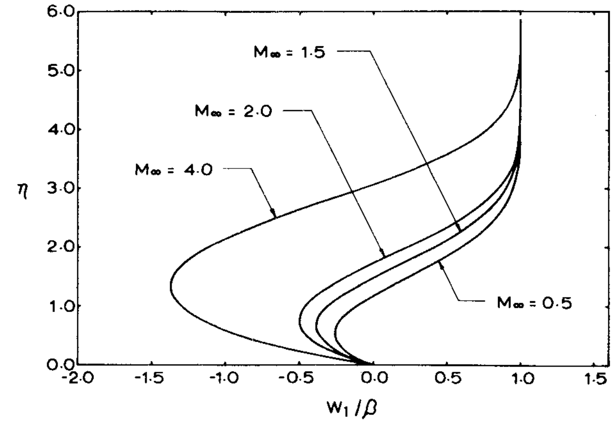

Fig. 3c Asymptotic cross flow velocity, $T_w = 0.50$.

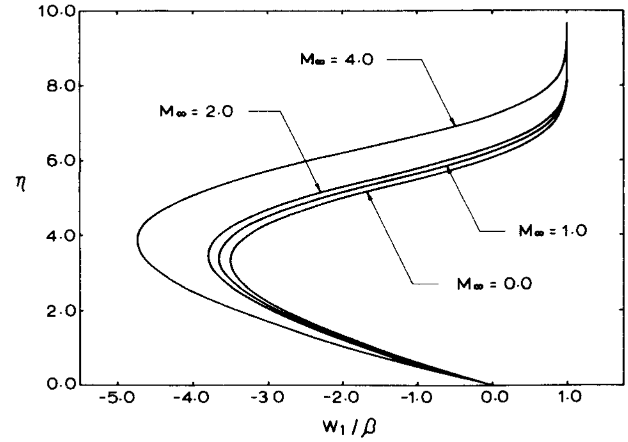

Fig. 3e Asymptotic cross flow velocity, $T_w = 4.0$.

Using Eq. (26) and defining the similarity variables as

$$\eta = \frac{Y}{(2x)^{1/2}}, \quad v_1(x, Y) = \frac{1}{(2x)^{1/2}} v_1(\eta), \quad \rho_o(x, Y) = \rho_o(\eta),$$

$$u_o(x, Y) = u_o(\eta), \quad w_1(x, Y) = \frac{1}{(2x)^{1/2}} w_1(\eta), \quad \mu_o(x, Y) = \mu_o(\eta)$$

Eq. (25) reduces to the following equation:

$$\mu_o \frac{d^2 w_1}{d\eta^2} + \left[\frac{d\mu_o}{d\eta} + \rho_o(\eta u_o - v_1)\right]\frac{dw_1}{d\eta} + \rho_o u_o w_1 = \beta \tag{27}$$

The boundary conditions for $w_1(\eta)$ require that

$$w_1(0) = 0 \quad \text{and} \quad w_1(\eta) \to \beta \quad \text{as} \quad \eta \to \infty \tag{28}$$

It is clear, from Eq. (27), that calculation of the cross flow $w_1(\eta)$ requires prior solution of the classical boundary-layer flow. This solution has been obtained earlier by Van Driest[11] for a general compressible fluid. However, it is a numerical solution and, therefore, had to be regenerated within the computer for use in solving Eq. (27) for the cross flow. The second order differential equations for classical boundary layer as well as for the cross flow velocity w_1 were solved using an implicit finite difference technique. An IBM 370/165 computer was used for the numerical computations and all results presented here were obtained in 30 sec of computing time.

The classical boundary-layer results, using nonunity Prandtl number and Sutherland's viscosity law, were found to agree well with those obtained by Van Driest.[11] The solutions for the cross flow $w_1(\eta)$ were compared, for the case of a model fluid, with the results of Libby presented in Refs. 3 and 5. Figure 2a shows the comparison for the cases when there is no heat transfer across the solid wall, whereas Fig. 2b represents the comparison for the cases with prescribed wall temperatures. Both comparisons show complete agreement of the present results with those obtained by Libby. The assumption of the model fluid was then removed and the results calculated using Pr = 0.718 and Sutherland's viscosity law. As Figs. 2a and 2b show, the cross flow results for the general compressible fluid deviate from the results for the model fluid, as the freestream Mach number increases.

The effects of compressibility and wall temperature boundary condition on the cross flow are presented in Figs. 3a–3e. The results for the adiabatic wall are shown in Fig. 3a where M_∞ ranges from almost zero up to a value of 4.0. Increasing Mach number is seen to increase the extent of the region as well as the magnitude of the reversed cross flow in the boundary layer. For the cases presented in Figs. 3b–3e, where the wall temperature is prescribed as T_w, an increase of T_w also results in amplifying the reversal of the secondary flow in the boundary layer, for all Mach numbers analyzed.

Figure 4 presents the wall shear stress coefficient due to the cross flow $w_1(\eta)$. It is seen that, whereas the classical boundary layer skin friction coefficient decreases with increase in the Mach number, the magnitude of the cross flow skin friction coefficient increases with increasing M_∞. Further, the classical boundary-layer shear stress coefficient at the wall, with a prescribed wall

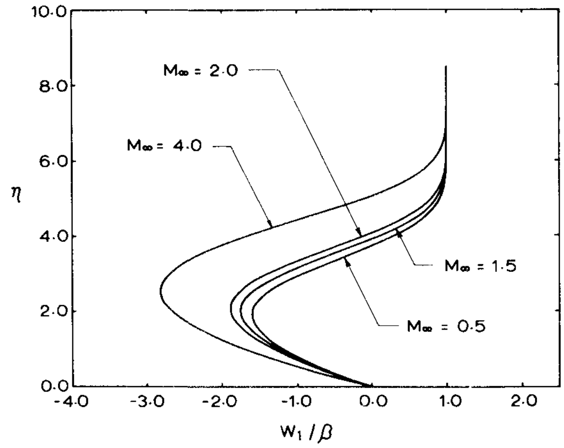

Fig. 3d Asymptotic cross flow velocity, $T_w = 2.0$.

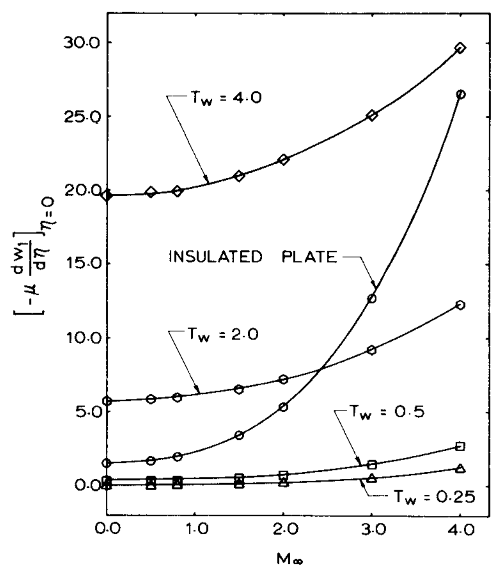

Fig. 4 Cross flow skin-friction parameter.

temperature T_w, decreases as T_w is increased; on the other hand, the magnitude of the wall shear due to the cross flow increases with an increase in T_w. The contrary behavior of the cross flow skin friction parameter may be explained as follows. The cross flow component W_1 of the potential velocity is positive at $z = 0$ as seen from Eqs. (9), (15) and (24) for incompressible and compressible flows. But the pressure gradient for the cross flow becomes adverse as shown in Eq. (26) and is directly proportional to the magnitude of the displacement thickness parameter β, which increases with increase of M_∞. As the displacement thickness becomes larger, the classical wall shear coefficient $\mu_o(\partial u_o/\partial \eta)_{\eta=0}$ decreases. However, the increased adverse pressure gradient for the cross flow causes an increased reversal of the cross flow and hence larger cross flow skin-friction coefficient.

Conclusion

The zeroth and the first-order inviscid flow along an axial corner has been determined for a general compressible fluid. This solution enables determination of not only the classical boundary-layer flow on the flat plates comprising the corner, but also of the secondary cross flow in the boundary layer. This solution is needed in prescribing the far field boundary condition [as $\zeta \to \infty$] for the viscous flow in the corner layer region, since the differential equations governing the corner layer flow are elliptic in the cross plane (y, z) or the similarity plane (η, ζ). One would expect that, as $\zeta \to \infty$, the corner layer solution should simply approach the classical boundary-layer solution. However, the Cartesian coordinate system used is not an optimal coordinate system for the problem, so that the cross flow w continues to persist even as $\zeta \to \infty$. This is the three-dimensional analog of the familiar 2-D boundary-layer flow over a flat plate where, in Cartesian coordinates, the normal velocity v persists even as $\eta \to \infty$. The use of optimal coordinates, e.g., parabolic coordinates for flow past a flat plate, is helpful in overcoming such anomalies.[12] Optimal coordinates for the three-dimensional flow along the axial corner have not been determined yet; the authors are presently pursuing work in this direction. But so long as Cartesian coordinates are used to formulate the corner flow problem, it is necessary to employ the cross flow $w_1(\eta)$ at the boundary $\zeta \to \infty$.

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